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Fuzzy Algebraic Theories

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The correspondence between (finitary) algebraic theories and (finitary) monads is well established since the 60s (see, for instance Hyland and Power 2007; Linton 1966; Lawvere 1963; Robinson 2002; Barr and Wells 2000; Manes 2012; Adámek, Rosickỳ, and Vitale 2010).

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Monads (and Lawvere Theories) are defined on arbitrary categories, while it seems difficult to build syntax for algebraic reasoning in categories different from **Set**.

Example: Quantitative algebras

An example of a solution of such problem for the category of extended metric spaces is given by the work of Mardare, Bacci, Plotkin and Panangaden on quantitative algebras Bacci et al. 2018; Mardare, Panangaden, and Plotkin 2017; Mardare, Panangaden, and Plotkin 2016.
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Main aim

Our main aim is to deal with this problem for **fuzzy sets**.

Let's start with an introduction to the category of fuzzy sets.

Definition (Wyler 1991; Wyler 1995)

Let H be a frame. A *H-fuzzy set* is a pair (A, μ_A) consisting in a set A and a *membership function* $\mu_A : A \to H$.
An arrow $f : (A, \mu_A) \to (B, \mu_B)$ is a function $f : A \to B$ such that $\mu_A(x) \leq \mu_B(f(x))$ for all $x \in A$.

In this way we get a category $\mathbf{Fuz}(H)$.

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Introducing fuzzy sets

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- The forgetful functor into **Set** has both right and left adjoint, taking as μ the constant function at $\top$ and $\bot$ respectively.
- $\mathbf{Fuz}(H)$ has all products, given by the products of set endowed with the pointwise infimum of the membership degrees of the components;
- More generally, the forgetful functor $\mathbf{Fuz}(H) \to \mathbf{Set}$ is topological, so $\mathbf{Fuz}(H)$ is complete.

Some examples of fuzzy algebraic structures (see Mordeson, Malik, and Kuroki 2012; Rosenfeld 1971; Ajmal 1994; Ajmal and Prajapati 1992; Mashour, Ghanim, and Sidky 1990).

Ideals

Consider a pair $((A, \mu), \cdot)$ of a fuzzy set and a function $\cdot : A \times A \to A$ such that $(A, \cdot)$ is a semigroup, we say that $((A, \mu), \cdot)$ is an *ideal* if, for every $x, y \in A$:

$$\mu(y) \leq \mu(x \cdot y) \text{ and } \mu(x) \leq \mu(x \cdot y)$$

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Normal groups

Consider a pair $((A, \mu), \cdot)$ of a fuzzy set and a function $\cdot : A \times A \to A$ such that $(A, \cdot)$ is a group, we say that $((A, \mu), \cdot)$ is *normal* if, for every $x, y \in A$,

$$\mu(x) \leq \mu(y \cdot x \cdot y^{-1})$$

Syntax: grammar

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- Two kinds of formulae:
  - *equations* are just pairs of terms, written as $t \equiv s$
  - *membership propositions* are pairs given by an element of H and a term, denoted by $\mathsf{E}(h, t)$.

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- A *sequent* $\Psi \vdash \phi$ is a pair (Ψ, ϕ) given by a set Ψ of formulae and a single formula ϕ.
- A *fuzzy algebraic theory* is simply a set of sequents on the same signature.

The sequent calculus we propose for algebraic reasoning is given by the following rules:

$$\frac{\phi \in \Gamma}{\Gamma \vdash \phi}\ \text{A} \qquad \frac{\Gamma \vdash \phi}{\Gamma \cup \Delta \vdash \phi}\ \textsc{Weak} \qquad \frac{\{\Gamma \vdash \phi \mid \phi \in \Phi\} \qquad \Phi \vdash \psi}{\Gamma \vdash \psi}\ \textsc{Cut}$$

$$\frac{}{\Gamma \vdash s \equiv s}\ \textsc{Refl} \qquad \frac{\Gamma \vdash s \equiv t}{\Gamma \vdash t \equiv s}\ \textsc{Sym} \qquad \frac{\Gamma \vdash s \equiv t \qquad \Gamma \vdash t \equiv u}{\Gamma \vdash t \equiv u}\ \textsc{Trans}$$

$$\frac{\sigma : X \to \mathcal{L}\text{-Terms} \qquad \Gamma \vdash \psi}{\Gamma[\sigma] \vdash \psi[\sigma]}\ \textsc{Sub} \qquad \frac{f \in O \qquad \{\Gamma \vdash t_i \equiv s_i\}_{i=1}^{\mathsf{ar}(f)}}{\Gamma \vdash f\left(t_1, \dots, t_{\mathsf{ar}(f)}\right) \equiv f\left(s_1, \dots, s_{\mathsf{ar}(f)}\right)}\ \textsc{Cong}$$

$$\frac{}{\Gamma \vdash \mathsf{E}(\bot, t)}\ \textsc{Inf} \qquad \frac{\Gamma \vdash \mathsf{E}(l, t)}{\Gamma \vdash \mathsf{E}(l \wedge l', t)}\ \textsc{Mon} \qquad \frac{\{\Gamma \vdash \mathsf{E}(l_i, t_i)\}_{i=1}^{\mathsf{ar}(f)}}{\Gamma \vdash \mathsf{E}\left(\inf\left(\{l_i\}_{i=1}^{n}\right), f\left(t_1, \dots, t_{\mathsf{ar}(f)}\right)\right)}\ \textsc{Exp}$$

$$\frac{S \subset H \qquad \{\Gamma \vdash \mathsf{E}(l, t)\}_{l \in S}}{\Gamma \vdash \mathsf{E}(\sup(S), t)}\ \textsc{Sup} \qquad \frac{\Gamma \vdash t \equiv s \qquad \Gamma \vdash \mathsf{E}(l, t)}{\Gamma \vdash \mathsf{E}(l, s)}\ \textsc{Fun}$$

Derivability is defined in the usual way.

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Ideals, II

Let Σ_S be the signature of semigroups, the theory of *ideal* Λ_I is the usual theory of semigroups plus the axioms (for each $h \in H$)

$$\mathsf{E}(h, y) \vdash \mathsf{E}(h, x \cdot y) \qquad \mathsf{E}(h, x) \vdash \mathsf{E}(h, x \cdot y)$$

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Groups, II

Let Σ_G be the signature of groups, the theory Λ_N of normal fuzzy groups is the usual theory of groups to which we add the axiom:

$$\mathsf{E}(l, x) \vdash \mathsf{E}\Big(l, y \cdot (x \cdot y^{-1})\Big)$$

Now that we have a syntax, we can give a semantics for our sequent calculus. Start with a signature $\Sigma = (O, C, \mathsf{ar})$, we can define Σ-*algebras* in the usual way: they are given by a fuzzy set (A, μ_A) endowed with a collection of arrows:

$$[\![\sigma]\!] : (A, \mu_A)^{\mathsf{ar}(f)} \to (A, \mu_A) \qquad [\![c]\!] : (1, \bot) \to (A, \mu_A)$$

Morphisms of Σ-*algebras* are simply morphisms of $\mathbf{Fuz}(H)$ which commutes with operations and constants.
In this way we get a category $\mathbf{Alg}(\Sigma)$.

Let $\mathcal{L} = (\Sigma, X)$ be a language and $((A, \mu_A), \{[\![\sigma]\!]\})$, then for every function $\iota : X \to A$ we can define the interpretation in A of all terms of language $\mathscr{L}$ with respect to ι.

Definition

A Σ-algebra $((A, \mu), \{[\![\sigma]\!]\})$ *satisfies a formula* ϕ *with respect to* ι $(((A, \mu_A), \{[\![\sigma]\!]\}) \vDash_\iota \phi)$, if

- if ϕ is $\mathsf{E}(h, t)$ then $h \leq \mu([\![t]\!])$
- if ϕ is $t \equiv s$ then $[\![t]\!] = [\![s]\!]$.

If this is true for every ι we say that the algebra *satisfies* ϕ.
Satisfiability of sequents is defined in the usual way.
We will write $\mathbf{Mod}(\Lambda)$ for the full subcategory of $\mathbf{Alg}(\Sigma)$ algebras which satisfy all the axioms of Λ.

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It's easy to see that the theories Λ_I, and Λ_N correspond to the categories of ideals and of normal fuzzy groups.

Soundness and completeness

Dealing with a bit of the usual bureaucracy it is possible to show the following two theorems.

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If a Σ-algebra satisfies all the premises of a rule of the fuzzy sequent calculus then it satisfies also its conclusion.

Theorem (completeness for formulae)

For any theory Λ, a formula ϕ is satisfied by all algebras in $\mathbf{Mod}(\Lambda)$ if and only if $\vdash \phi$ is derivable from Λ.

The free model of a theory

Given a theory Λ in the language (Σ, X) we will now provide a recipe for the free model F_Λ on a given fuzzy set (A, μ).

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- Construct the signature Σ' adding to Σ a constant c_a for every $a \in A$, in this new language we can take the theory Λ' obtained adding to Λ the sequents $\vdash \mathsf{E}(h, a)$ where $a \in A$ and $\mu(a) \geq h$.

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- Construct a model of Λ taking the set of terms in the language (Σ', X) and quotienting it by the relation which identifies t with s if and only if $\vdash t \equiv s$ is derivable from Λ. Equip it with the function μ_Λ which sends
$$[t] \mapsto \sup\{h \in H \mid \vdash \mathsf{E}(h, t) \text{ is derivable from } \Lambda\}$$

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The previous construction yields a functor $\mathbf{Fuz}(H) \to \mathbf{Mod}(\Lambda)$ which is the left adjoint to the forgetful functor $U_\Lambda : \mathbf{Mod}(\Lambda) \to \mathbf{Fuz}(H)$.

T_Λ is the monad $U_\Lambda \circ F_\Lambda : \mathbf{Fuz}(H) \to \mathbf{Fuz}(H)$.

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Definition

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Theorem

For any basic theory Λ, $\mathbf{EM}(T_\Lambda)$ and $\mathbf{Mod}(\Lambda)$ are isomorphic, and thus equivalent, categories.

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Theorem (HSP-Theorem)

Let Σ be an algebraic signature. A class of algebras for Σ is the class of models for some theory if and only if it is closed under homomorphic images (H), subalgebras (S) and products (P).

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We now aim to obtain a HSP-theorem for our notion of Σ-algebras, using the machinery developed in (Milius and Urbat 2019).

To exploit Milius and Urbat's results we need some ingredients.
We fix a triple $(\mathbf{C}, (\mathscr{E}, \mathscr{M}), \mathscr{X})$ where $\mathbf{C}$ is a category with small products, $(\mathscr{E}, \mathscr{M})$ is a proper factorization system on it and $\mathscr{X}$ is a class of objects of $\mathbf{C}$.
These must satisfy the following two conditions:

- for any $X \in \mathscr{X}$, the class $X{\downarrow}\mathbf{C}$ of all $e \in \mathscr{E}$ with domain X is essentially small.
- for every object C of $\mathbf{C}$ there exists $e : X \to C$ in $\mathscr{E}_{\mathscr{X}}$ with $X \in \mathscr{X}$.

Remark

Here $\mathscr{E}_{\mathscr{X}}$ is the class of $e : A \to B \in \mathscr{E}$ such that for every $X \in \mathscr{X}$, X is *projective* with respect to e ($e_* : \mathbf{C}(X, A) \to \mathbf{C}(X, B)$ is surjective.)

Definition

An $\mathscr{X}$-*equation* is an arrow $e \in X{\downarrow}\mathbf{C}$ with $X \in \mathscr{X}$. We say that an object A of $\mathbf{C}$ *satisfies* $e : X \to C$, and we write $A \vDash_{\mathscr{X}} e$, if for every $h : X \to A$ there exists $q : C \to A$ such that $q \circ e = h$.

Given a class $\mathbb{E}$ of $\mathscr{X}$-equations, we define $\mathcal{V}(\mathbb{E})$ as the full subcategory of $\mathbf{C}$ given by objects that satisfy e for every $e \in \mathbb{E}$.

A full subcategory $\mathbf{V}$ is $\mathscr{X}$-*equationally presentable* if there exists $\mathbb{E}$ such that $\mathbf{V} = \mathcal{V}(\mathbb{E})$.

Urbat and Milius' machinery: main result

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Theorem (Milius and Urbat 2019, Th. 3.15, 3.16)

A full subcategory $\mathbf{V}$ of $\mathbf{C}$ is $\mathscr{X}$-equationally presentable if and only if it is closed under $\mathscr{E}_{\mathscr{X}}$-quotients, $\mathscr{M}$-subobjects and (small) products.

To apply the previous theorem we take $\mathbf{C}$ to be $\mathbf{Alg}(\Sigma)$ for some fuzzy signature Σ, this has a factorization system given by

$$\mathscr{E}_\Sigma = \{e \in \mathbf{Alg}(\Sigma) \mid U_\Sigma(e) \text{ is epi}\}$$

$$\mathscr{M}_\Sigma = \{m \in \mathbf{Alg}(\Sigma) \mid U_\Sigma(m) \text{ is a strong mono}\}$$

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Then we take the following two classes:

$$\mathscr{X}_0 = \{\mathscr{F}_\Sigma(X, \mu) \mid \mu(x) = \bot \text{ for every } x \in X\}$$

$$\mathscr{X}_{\mathsf{E}} = \{\mathscr{F}_\Sigma(X, \mu_X) \mid (X, \mu_X) \in \mathbf{Fuz}(H)\}$$

Application to fuzzy algebraic theories

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Lemma

We have that

$$\mathscr{E}_{\Sigma,\mathscr{X}_0} = \mathscr{E}_\Sigma \quad \mathscr{E}_{\Sigma,\mathscr{X}_{\mathsf{E}}} = \{e \in \mathscr{E}_\Sigma \mid \mathscr{U}_\Sigma(e) \text{ splits}\}$$

Moreover $(\mathbf{Alg}(\Sigma), (\mathscr{E}_\Sigma, \mathscr{M}_\Sigma), \mathscr{X}_0)$ and $(\mathbf{Alg}(\Sigma), (\mathscr{E}_\Sigma, \mathscr{M}_\Sigma), \mathscr{X}_{\mathsf{E}})$ both satisfy the assumptions of (Milius and Urbat 2019).

We want now to translate formulae of our sequent calculus into $\mathscr{X}_0$- and $\mathscr{X}_{\mathsf{E}}$-equations.

Definition

A theory Λ is said to be:

- *unconditional* if any sequent in Λ is of the form $\vdash \phi$ for some formula ϕ;
- *of type* E if any sequent in Λ is of the form $\{\mathsf{E}(l_i, x_i)\}_{i \in I} \vdash \phi$ for some formula ϕ, $\{x_i\}_{i \in I} \subset X$ and $\{l_i\}_{i \in I} \subset H$.

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Lemma

For any signature Σ and $\mathscr{X}_{\mathsf{E}}$-equation $e : F_\Sigma(X, \mu_X) \to \mathcal{B}$ there exists a theory Λ_e of type E such that, a Σ-algebra satisfies e if and only if it belongs to $\mathbf{Mod}(\Lambda_e)$.
Moreover $|\Gamma| \leq \left|\mu_X^{-1}(H \smallsetminus \{\bot\})\right|$ for any $\Gamma \vdash \phi \in \Lambda_e$.

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Corollary

For any signature Σ and $\mathscr{X}_0$-equation $e : F(X, \bot) \to \mathcal{B}$ there exists an unconditional theory Λ_e such that a Σ-algebra satisfies e if and only if it belongs to $\mathbf{Mod}(\Lambda_e)$.

Theorem

Let $\mathbf{V}$ be a full subcategory of $\mathbf{Alg}(\Sigma)$.
The following are equivalent:

- $\mathbf{V}$ is closed under epimorphisms, (small) products and strong monomorphisms
- there exists a class of unconditional theories $\{\Lambda_e\}_{e\in\mathbb{E}}$ such that a Σ-algebra belongs to $\mathbf{V}$ if and only if it belongs to $\mathbf{Mod}(\Lambda_e)$ for all $e \in \mathbb{E}$.

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Moreover, also the following are equivalent:

- $\mathbf{V}$ is closed under split epimorphisms, (small) products and strong monomorphisms
- there exists a class of type E theories $\{\Lambda_e\}_{e\in\mathbb{E}}$ such that a Σ-algebra belongs to $\mathbf{V}$ if and only if it belongs to $\mathbf{Mod}(\Lambda_e)$ for all $e \in \mathbb{E}$.

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Conclusions and Further work

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Further work to be done:

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- Arities for us are simply numbers; how can we reconcile this with the approach based on Lawvere theories, in which arities are given by finite fuzzy sets?

Conclusions and Further work

We have introduced a syntax which recovers the connection between algebraic theories and monads for fuzzy sets.
Moreover, we have proved a HSP theorem for two kinds of such theories.

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- Arities for us are simply numbers; how can we reconcile this with the approach based on Lawvere theories, in which arities are given by finite fuzzy sets?
- $\mathbf{Fuz}(H)$ may be not the best environment to do "fuzzy mathematics" (Pitts 1982). An alternative is given by the topos of H-sets (Fourman and Scott 1979). Is it possible to produce a syntax for algebraic theories in this new enviroment?

Notice that there is a size issue here: we cannot arrange the collection $\{\Lambda_e\}_{e\in\mathbb{E}}$ into a unique theory since a proper class of variables is needed to write down all the necessary sequents. This can be avoided if $\{\Lambda_e\}_{e\in\mathbb{E}}$ satisfies a boundedness property about the premises of the sequents belonging to each Λ_e.

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Definition

Given a cardinal κ we say that a $\mathscr{X}_{\mathsf{E}}$-equation $e : F_\Sigma(X, \mu_X) \to \mathcal{B}$ is *κ-supported* if $|\mathsf{supp}(X, \mu_X)| < \kappa$.

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Proposition

Let $\mathbf{V} = \mathcal{V}(\mathbb{E})$ be an $\mathcal{X}_{\mathsf{E}}$-equational defined subcategory of $\mathbf{Alg}(\Sigma)$ and suppose every $e \in \mathbb{E}$ is κ-supported, then there exists a theory Λ in the language (Σ, κ), such that $\mathbf{V} = \mathbf{Mod}(\Lambda)$.

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Corollary

$\mathbf{V}$ is closed under epimorphisms, (small) products and strong monomorphisms if and only if there exists a language $\mathcal{L}$ and an unconditional theory Λ such that $\mathbf{V} = \mathbf{Mod}(\Lambda)$.