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A new criterion for M, N-adhesivity

with an application to hierarchical graphs

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Roughly speaking an adhesive category is a category in which pushouts along monos interact with pullbacks in the same way as they do in a topos (Lack and Sobocinski 2006; Garner and Lack 2012; Johnstone, Lack, and Sobocinski 2007).

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In this framework one can prove in an abstract way results such the Local Church-Rosser Theorem or the Concurrency Theorem (see Lack and Sobociński 2005).

Adhesivity is a property of pushout squares having a mono as one of the two given sides, so we can generalize this notion with two further steps:

- ($\mathcal{M}$-adhesivity see Azzi, Corradini, and Ribeiro 2019) ask the adhesivity property for squares in which one of the given sides comes from a suitable class $\mathcal{M}$ of monos;
- ($\mathcal{M}$, $\mathcal{N}$-adhesivity see Habel and Plump 2012) ask the adhesivity property for squares in which one of the given sides comes from a suitable class $\mathcal{M}$ of monos and the other comes from another suitable class $\mathcal{N}$.

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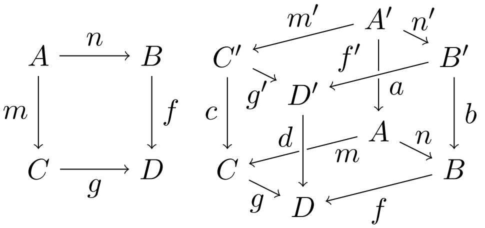
A pushout square as the one on the left below in a category is *Van Kampen* if in any cube constructed upon it, having pullbacks as back faces, the top face is a pushout if and only if the front faces are pullbacks.



$\mathcal{M}, \mathcal{N}$-adhesivity: definition

Now, take a category $\mathbf{A}$, and fix a class of monos $\mathcal{M}$ and a class of arrows $\mathcal{N}$. Suppose also that they interact "nicely" (i.e. they enjoy some composition and decomposition property).

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Definition (Habel and Plump 2012)

Given $\mathbf{A}$, $\mathcal{M}$ and $\mathcal{N}$ as above, $\mathbf{A}$ is $\mathcal{M}$, $\mathcal{N}$-*adhesive* if

1. every cospan $C \xrightarrow{g} D \xleftarrow{m} B$ with $m \in \mathcal{M}$ can be completed to a pullback ($\mathcal{M}$-*pullbacks*);
2. every span $C \xleftarrow{m} A \xrightarrow{n} B$ with $m \in \mathcal{M}$ and $n \in \mathcal{N}$ can be completed to a pushout; ($\mathcal{M},\mathcal{N}$-*pushouts*);
3. $\mathcal{M},\mathcal{N}$-pushouts are Van Kampen squares.

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3. $\mathcal{M},\mathcal{N}$-pushouts are Van Kampen squares.

Taking $\mathcal{M}$ to be the class of all (regular) monos and $\mathcal{N}$ to be the class of all maps we get back the usual notion of (quasi)adhesivity.

In many cases the proof of $\mathcal{M},\mathcal{N}$-adhesivity of a category **A** is given shifting the calculus of pullbacks and pushouts to another **B** whose adhesivity properties are known. Our criterion is a formalization of this procedure.

A simple criterion

In a lot of cases the proof of $\mathcal{M},\mathcal{N}$-adhesivity of a category $\mathbf{A}$ is given shifting the calculus of pullbacks and pushouts to another $\mathbf{B}$ whose adhesivity properties are known. Our criterion is a formalization of this procedure. We start with a technical definition.

Definition

Let $I : \mathbf{I} \to \mathbf{C}$ be a diagram and J a set. We say that a family $F = \{F_j\}_{j\in J}$ of functors $F_j : \mathbf{C} \to \mathbf{D}_j$

- *jointly preserves (co)limits* of I if given a (co)limiting (co)cone $(L, l_i)_{i\in\mathbf{I}}$ for I, every $(F_j(L), F_j(l_i))_{i\in\mathbf{I}}$ is (co)limiting for $F_j \circ I$;

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- *jointly reflects (co)limits* of I if a (co)cone $(L, l_i)_{i\in\mathbf{I}}$ is (co)limiting for I whenever $(F_j(L), F_j(l_i))_{i\in\mathbf{I}}$ is (co)limiting for $F_j \circ I$ for every $j \in J$;

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Let $I : \mathbf{I} \to \mathbf{C}$ be a diagram and J a set. We say that a family $F = \{F_j\}_{j\in J}$ of functors $F_j : \mathbf{C} \to \mathbf{D}_j$

- *jointly lifts (co)limits* of I if given a (co)limiting (co)cone $(L_j, l_{j,i})_{i\in\mathbf{I}}$ for every $F_j \circ I$, there exists a (co)limiting (co)cone $(L, l_i)_{i\in\mathbf{I}}$ for I such that $(F_j(L), F_j(l_i))_{i\in\mathbf{I}} = (L_j, l_{j,i})_{i\in\mathbf{I}}$ for every $j \in J$.

A simple criterion

In a lot of cases the proof of $\mathcal{M},\mathcal{N}$-adhesivity of a category $\mathbf{A}$ is given shifting the calculus of pullbacks and pushouts to another $\mathbf{B}$ whose adhesivity properties are known. Our criterion is a formalization of this procedure. We start with a technical definition.

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Let $I : \mathbf{I} \to \mathbf{C}$ be a diagram and J a set. We say that a family $F = \{F_j\}_{j\in J}$ of functors $F_j : \mathbf{C} \to \mathbf{D}_j$

- *jointly creates (co)limits* of I if I has a (co)limit and F jointly preserves and reflects (co)limits along it.

A simple criterion

We are now ready to state our criterion. We will fix a category $\mathbf{A}$, classes $\mathcal{M}$ and $\mathcal{N}$ and a non empty family of functors $F_j : \mathbf{A} \to \mathbf{B}_j$ such that $\mathbf{B}_j$ is $\mathcal{M}_j, \mathcal{N}_j$-adhesive and $F_j(\mathcal{M}) \subset \mathcal{M}_j$, $F_j(\mathcal{N}) \subset \mathcal{N}_j$ for every $j \in J$. We can prove the following:

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Theorem

1) If every F_j preserves pullbacks, F jointly preserves $\mathcal{M}, \mathcal{N}$-pushouts, and jointly reflects pushout squares as aside with $m, n \in \mathcal{M}$ and $f \in \mathcal{N}$, $\mathcal{M}$-pullbacks and $\mathcal{N}$-pullbacks then $\mathcal{M}, \mathcal{N}$-pushouts are Van Kampen squares.

$$\begin{array}{ccc} F_j(A) & \xrightarrow{F_j(f)} & F_j(B) \\ {\scriptstyle F_j(m)}\downarrow & & \downarrow{\scriptstyle F_j(n)} \\ F_j(C) & \xrightarrow[F_j(g)]{} & F_j(D) \end{array}$$

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Theorem

2) If F satisfies the assumptions of point 1) and jointly creates both $\mathcal{M}$-pullbacks and $\mathcal{N}$-pullbacks, then $\mathbf{A}$ is $\mathcal{M}, \mathcal{N}$-adhesive.

A simple criterion

We are now ready to state our criterion. We will fix a category $\mathbf{A}$, classes $\mathcal{M}$ and $\mathcal{N}$ and a non empty family of functors $F_j : \mathbf{A} \to \mathbf{B}_j$ such that $\mathbf{B}_j$ is $\mathcal{M}_j, \mathcal{N}_j$-adhesive and $F_j(\mathcal{M}) \subset \mathcal{M}_j$, $F_j(\mathcal{N}) \subset \mathcal{N}_j$ for every $j \in J$. We can prove the following:

Theorem

3) If F jointly creates all pushouts and all pullbacks, then $\mathbf{A}$ is $\mathcal{M}_F, \mathcal{N}_F$-adhesive, where

$$\mathcal{M}_F := \{m \in \mathbf{A} \mid F_j(m) \in \mathcal{M}_j \text{ for every } j \in J\}$$
$$\mathcal{N}_F := \{n \in \mathbf{A} \mid F_j(n) \in \mathcal{N}_j \text{ for every } j \in J\}$$

A first application of our criterion is given by comma categories. Recall that the *comma category* $L{\downarrow}R$ of $L : \mathbf{A} \to \mathbf{C}$, $R : \mathbf{B} \to \mathbf{C}$ has arrows $f : L(A) \to R(B)$ as objects and squares as the one aside as morphisms.

$$\begin{array}{ccc} L(A) & \xrightarrow{L(h)} & L(A') \\ f \downarrow & & \downarrow g \\ R(C) & \xrightarrow[R(k)]{} & R(C') \end{array}$$

Application: comma categories

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$$\begin{array}{ccc} L(A) & \xrightarrow{L(h)} & L(A') \\ f\downarrow & & \downarrow g \\ R(C) & \xrightarrow[R(k)]{} & R(C') \end{array}$$

Theorem

Let $I : \mathbf{I} \to L{\downarrow}R$ be a diagram such that L preserves the colimit (if it exists) of $U_L \circ I$. Then the two forgetful functors $U_L : L{\downarrow}R \to \mathbf{C}$, $U_R : L{\downarrow}R \to \mathbf{C}$ jointly creates colimits of I.

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$$\begin{array}{ccc} L(A) & \xrightarrow{L(h)} & L(A') \\ f\downarrow & & \downarrow g \\ R(C) & \xrightarrow[R(k)]{} & R(C') \end{array}$$

Corollary

Let $\mathbf{A}$ and $\mathbf{B}$ be respectively $\mathcal{M},\mathcal{N}$-adhesive and $\mathcal{M}',\mathcal{N}'$-adhesive categories, $L : \mathbf{A} \to \mathbf{C}$ a functor that preserves $\mathcal{M},\mathcal{N}$-pushouts, and $R : \mathbf{B} \to \mathbf{C}$ a pullback preserving one. Then $L{\downarrow}R$ is $\mathcal{M}{\downarrow}\mathcal{M}',\mathcal{N}{\downarrow}\mathcal{N}'$-adhesive, where

$$\mathcal{M}{\downarrow}\mathcal{M}' := \{(h,k) \in L{\downarrow}R \mid h \in \mathcal{M}, k \in \mathcal{M}'\}$$
$$\mathcal{N}{\downarrow}\mathcal{N} := \{(h,k) \in L{\downarrow}R \mid h \in \mathcal{N}, k \in \mathcal{N}'\}.$$

Another important example is that of *hierarchical graphs* (Palacz 2004; Padberg 2017). Roughly speaking hierarchical graphs are graphs in which the set of edges comes equipped with some structure. Usually with also want a graph to display a *interface*, i. e. a given subset of its nodes.

Hierarchical graphs

Another important example is that of *hierarchical graphs* (Palacz 2004; Padberg 2017). Roughly speaking hierarchical graphs are graphs in which the set of edges comes equipped with some structure. Usually with also want a graph to display a *interface*, i. e. a given subset of its nodes.

Formally, take the forgetful functor $L : \mathbf{S} \to \mathbf{Set}$ from some category of *structured sets* and the functor $R : \mathbf{Set}^2 \to \mathbf{Set}$ which sends $f : X \to Y$ to $Y \times Y$. Then the category of graphs in which the set of edges is an object of $\mathbf{S}$ is simply $[L, R]$.

Hierarchical graphs

From our criterion it follows that, if L is well-behaved with respect to pushouts, the category of hierarchical graphs inherits the adhesivity properties of $\mathbf{S}$.

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We can replace graphs with *hyper*graph using thee functor $\mathbf{Set}^2 \to \mathbf{Set}$ which sends $f : X \to Y$ to the free monoid on Y (this relies on the fact that the free monoid monad is cartesian and thus preserves pullbacks, see Carboni and Johnstone 1995).

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2. We can take as **S** the category od directed acyclic graphs. This category has the $\mathcal{M}, \mathcal{N}$-adhesivity property where
   - $\mathcal{M}$ is the class of *downward closed morphisms*: if a node is in the image of such a morphism then all its predecessors are in the image too;
   - $\mathcal{N}$ is the class of monos.

   Our criterion now yields immediately an adhesivity property for this kind of hierarchical (hyper)graphs.

Term graphs

Our last example is given by *term graphs* (Corradini and Gadducci 2005).

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Definition

Let $\Sigma = (O, \mathsf{ar})$ be an algebraic signature, *term graph over* Σ is given by a set V is a set, and two partial functions $l : V \rightharpoonup O$, $s : V \rightharpoonup V^\star$

- $\mathrm{dom}(l) = \mathrm{dom}(s)$;
- for each $v \in \mathrm{dom}(l)$, $\mathsf{ar}(l(v)) = \mathsf{length}(s(a))$, where $\mathsf{length} : V^\star \to \mathbb{N}$ associates to each word its length.

A morphism $(V, l, s) \to (W, t, r)$ is a function $f : V \to W$ such that, for every $v \in \mathrm{dom}(l)$.

$$t(f(v)) = l(v) \qquad r(f(v)) = f^\star(s(v))$$

Term graphs

Our last example is given by *term graphs* (Corradini and Gadducci 2005).

We can apply our criterion to the forgetful functor $\mathbf{TG}_\Sigma \to \mathbf{Set}$, allowing us to recover the following theorem.

Theorem

The category $\mathbf{TG}_\Sigma$ of term graphs is quasiadhesive.

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We can apply our criterion to the forgetful functor $\mathbf{TG}_\Sigma \to \mathbf{Set}$, allowing us to recover the following theorem.

Theorem

The category $\mathbf{TG}_\Sigma$ of term graphs is quasiadhesive.

This result was already known, but our framework allows us to dissect the traditional brute-force proof (see Corradini and Gadducci 2005) and to see how, instead, it relies on general and abstract facts.

Conclusions and further work

We have introduced a new criterion for $\mathcal{M}, \mathcal{N}$-adhesivity, abstracting from many *ad hoc* proofs found in literature. This criterion allows us to prove in a uniform and systematic way some previous results about the adhesivity of categories built by products, exponents, and comma construction, such as hierarchical (hyper)graphs and term graphs.

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Further work:

- we plan to analyse other categories of graph-like objects such as *(directed bi-graphs)* (Grohmann and Miculan 2007);

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Further work:

- we plan to analyse other categories of graph-like objects such as *(directed bi-graphs)* (Grohmann and Miculan 2007);
- verify if the $\mathcal{M}, \mathcal{N}$-adhesivity that we obtain from our criterion is suited for modelling specific rewriting systems: for instance $\mathbf{TG}_{\Sigma}$ is quasiadhesive but this does not suffice in most applications, because the rules are often spans of monomorphisms, and not of regular monos.

Thank you for your attention!

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